TRANSIENT ANALYSIS OF A TWO-UNIT SYSTEM MODELLED BY A GENERAL MARKOV PROCESS

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Abstract

A two-unit system with operational and repair times following phase-type distributions and incorporating geometrical processes is considered. A transient analysis is performed.

1 Introduction

It is frequent in the study of dynamic systems modellized by Markov processes to consider the steady-state because of calculations simplicity. The transient behavior is not so broadly studied, possibly due to the intractable expressions appearing in calculations. When only exponential times are involved some authors have considered the transient regime for studying some particular problems. However, the assumptions of exponential times sometimes is not practical. Recently, the phase type distributions have been introduced for operating and repair times. They extend the exponential family and preserve the Markovian structure in the model. A work in this line is Neuts et al. (2000), which considered a general Markov process for modelling a reliability system which can suffer external and internal failures, the first ones can be repairable or non-repairable, and the last one non-repairable; the stationary regime was studied. Pérez-Ocón et al. (2004a) studied that model in transient regime with operational times partitioned into two well-distinguished classes successively occupied: good and preventive. Pérez-Ocón et al. (2004b) studied the maintenance of a multiple system governed by a quasi-birth-and-death process. In the present work we extend the transient analysis results from an unit-system (Pérez-Ocón et al., 2004a) to a two-system in an natural way. For this system, the transition probabilities and some performance measures are explicitly calculated using an algorithmic approach that involves the Kronecker product. Moreover, an approach to the stationary regime is considered.

1.1 Model description

We will assume a system with two independent units. The two units undergo accidental failures following a Poisson process with rate λ, these failures can be repairable with probability p and non-repairable with probability $1-p$. Also, they are exposed to wear-out failure non-repairable. All failures are independent. If a non-repairable failure arrives, the unit is replaced by a new and identical one. There are two repairmen. Each unit has an operational time governed by a phase-type distribution, and these successive random times form a geometric process. The same for the repairing times. Each unit is replaced after a prefixed number of failures N. All times are independent random variables.

This is a general system, and it has been studied for an unit in stationary regime in Neuts et al. (2000), and in transient regime in Pérez-Ocón et. al (2004a). For sake of simplicity we consider N=1.

The probability distributions of the times to failure and repair times are all of phase-type with the following representations, where the subindex indicates the order of the matrices and the factor a_k, k = 1, 2, the deteriorating or growing reliability after the repair for unit k, depending on it will be greater or less than one.

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Table 1.1 Operational and repair times

	Time to failure after repair n (n = 0,1)	Repair time
Unit 1	$[\alpha(1), a_1^n T(1)]_{m1}$	$[\beta(1), S(1)]_{n_1}$
Unit 2	$[\alpha(2), a_2^n T(2)]_{m2}$	$[\beta(2), S(2)]_{n2}$

The unit k occupies one of these macro-states: new (macro-state 0), in repair (macro-state 1_R), or operational after a repair (macro-state 1). The states of the units are the phase occupied, so the macro-states $0, 1$ have m_k states and the macro-state 1_R has n_k states. When the system is considered, the macro-states are ordered pairs, corresponding to the units 1, 2: $\{(u, v); u, v = 0, 1, 1_R\}$. The state space E must be partitioned into four sets denoted by E_0, E_1, E_2,and E_3, where

$E_0 = \{(u, v); u, v = 0, 1\}$ has four macro-states $E_{01} = (0, 0), E_{02} = (0, 1)$, $E_{03} = (1, 0)$, $E_{04} = (1, 1)$, each one consisting in pairs (i, j) with $1 \leq i \leq m_1, 1 \leq j \leq m_2$;

$E_1 = \{(u, v); u = 0, 1; v = 1_R\}$ has two macro-states $E_{11} = (0, 1_R), E_{12} = (1, 1_R)$ consisting in pairs $\{(i, j); 1 \leq i \leq m_1, 1 \leq j \leq n_2\}$;

$E_2 = \{(u, v); u = 1_R; v = 0, 1\}$ has two macro-states E_{21}, E_{22} consisting in pairs $\{(i, j); 1 \leq i \leq n_1, 1 \leq j \leq m_2\}$; and

E_3 consists in pairs $\{(i, j); 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$.

The Markov process that governs the system has a state space E with the states (i, j) above describes. The generator will be construct using the macro-states structure:

$$
Q^S = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ E_3 \end{array}
\begin{array}{c}
\begin{array}{cccc} E_0 & E_1 & E_2 & E_3 \end{array} \\
\begin{pmatrix}
Q_1^S & Q_2^S & Q_3^S & 0 \\
Q_4^S & Q_5^S & 0 & Q_6^S \\
Q_7^S & 0 & Q_8^S & Q_9^S \\
0 & Q_{10}^S & Q_{11}^S & Q_{12}^S
\end{pmatrix}
\end{array}.
$$

The blocks of this matrix have been calculated and are available from the authors.

If both units are new, the initial probability vector is $[\alpha(1) \otimes \alpha(2), 0, 0, 0, 0, 0, 0, 0, 0]$.

2 Main results

2.1 Transient Probability Functions

The transition probability functions for the two-system depends on the ones for every unit. For an unit k, we denote by $P_\nu^k(i, j, t)$ the probability that the unit will be in the phase j of the operational macro-state i at time t, given that initially it occupies the phase v of the macro-state 0, with $k = 1, 2$, $i = 0, 1$, $1 \leq j \leq m_k$. Similarly, $P_\nu^k(i_R, \acute{j}, t)$ is the probability that the unit will be in the phase j' of the repairing state 1_R given the initial condition, for $\acute{j} = 1, .., n_k$.

The transition probabilities among the macro-state 0 and the operational macro-estate $i, i = 0, 1$,or the repairing macro-state 1_R are given respectively by the matrices:

$$P_i^k(t) = (P_\nu(i, j, t))_{\nu, j = 1, \ldots, m_k}$$

$$P_{1R}^k(t) = (P_\nu(i_R, \acute{j}, t))_{\substack{\nu = 1, \ldots, m_k \\ \acute{j} = 1, \ldots, n_k}}$$

The calculus of such matrices is given in Pérez-Ocón (2004a).

The transition probability functions for the system in terms of the corresponding in the units are given by

$$P^S(u, v, t) = P_u^1(t) \otimes P_v^2(t), \; u, v = 0, 1, 1_R$$

2.2 Performance measures

For this system the availability, the reliability in an interval, and the rate of occurrence of failures can be explicitly calculated. We obtain these measures for a parallel system.

2.2.1 Operational time

The random variable that denotes the lifetime of the system, T_S, follows a phase-type with representation (γ_S, L_S), being $\gamma_S = [\alpha(1) \otimes \alpha(2), 0, 0, 0, 0, 0, 0, 0]$ and

$$L_S = \begin{pmatrix} Q_1^S & Q_2^S & Q_3^S \\ Q_4^S & Q_5^S & 0 \\ Q_7^S & 0 & Q_8^S \end{pmatrix}.$$

2.2.2 Availability

The availability at time t is the probability that the system is operational at time $t, t \geq 0$. It results:

$$\begin{aligned} A_S(t) &= 1 - [\alpha(1) \otimes \alpha(2)] P_{E_3}^S(t) e_{n_1 n_2} \\ &= 1 - [\alpha(1) P_{1_R}^1(t) e_{n_1} \otimes \alpha(2) P_{1_R}^2(t) e_{n_2}] \\ &= 1 - \bar{A}_1(t) \bar{A}_2(t), \end{aligned}$$

$\bar{A}_k(t)$ being the probability that at time t the unit k is not operational, $k = 1, 2$. This can be written $A_S(t) = A_1(t)\ A_2(t)$.

2.2.3 Reliability

This is the probability that the system will be continuously operational up to time t, and it is easy to see that

$$\begin{aligned} R_S(t) &= \gamma_S \exp(L_S t) e = \\ &= 1 - \bar{R}_1(t) \bar{R}_2(t), \end{aligned}$$

$R_k(t)$ being the reliability of the unit k, $k = 1, 2$, with $\bar{R}_k(t) = 1 - R_k(t)$.

2.2.4 Reliability in the period $[t, t+\tau]$

The probability that system will be operational in the period $[t, t+\tau]$ is given by

$$\begin{aligned} R_S(t,\tau) &= (\alpha(1) \otimes \alpha(2)) [P_{E_{01}}^S(t), P_{E_{02}}^S(t), ..., P_{E_{22}}^S(t)] \exp(L_S \tau) e \\ &= R_S(t,\tau) = 1 - \bar{R}_1(t,\tau) \bar{R}_2(t,\tau). \end{aligned}$$

2.2.5 Rate of occurrence of failures (rocof)

The failure state of the system is E_3. Thus, the mean number of failures per unit time at time t can be expressed as

$$\begin{aligned} v_S(t) &= (\alpha(1) \otimes \alpha(2)) P_{E_{11}}^S(t) [T^0(1) \otimes e_{n_2}] + \\ &(\alpha(1) \otimes \alpha(2)) P_{E_{21}}^S(t) [e_{n_1} \otimes T^0(2)] \\ &= [\alpha(1) P_0^1(t) T^0(1))\ (\alpha(2) P_{1_R}^2(t) e_{n_2}) + \\ &(\alpha(1) P_{1_R}^1(t) e_{n_1})\ (\alpha(2) P_0^2(t) T^0(2)). \end{aligned}$$

2.3 Numerical Example

Consider a two-unit system under the assumptions of the general model. The parameter values of the accidental failures are: $\lambda = 0.00187$ and $p = 0.87$. The initial vectors for the operational and repair times of the units, that follow phase-type distribution, are $\alpha(1) = \alpha(2) = \beta(1) = \beta(2) = (1, 0, 0)$;

the operational times are governed by the matrices

$$T(1) = \begin{pmatrix} -0.0027 & 0.0027 & 0 \\ 0 & -0.008 & 0.008 \\ 0 & 0 & -0.02878 \end{pmatrix}; \, T(2) = \begin{pmatrix} -0.003 & 0.003 & 0 \\ 0 & -0.0075 & 0.0075 \\ 0 & 0 & -0.032 \end{pmatrix},$$

and the repair times by the matrices

$$S(1) = \begin{pmatrix} -0.02 & 0.02 & 0 \\ 0.01 & -0.08 & 0.07 \\ 0.005 & 0 & -0.1 \end{pmatrix}; \quad S(2) = \begin{pmatrix} -0.018 & 0.018 & 0 \\ 0.015 & -0.075 & 0.06 \\ 0.004 & 0 & -0.09 \end{pmatrix}.$$

After repair, the units modify its operational time by means of the factors: $a_1 = 1.25$ and $a_2 = 1.35$.

In Table 1.2 we show the probability functions for the system, being $O_{u,v}(t) = (\alpha(1) \otimes \alpha(2)) P^S_{(u,v)}(t)(e \otimes e)$. The performance measures for the parallel system are given in Table 1.3. The last row in the tables refers to the stationary regime. The corresponding results for each unit are given in Pérez-Ocón (2004a).

Table 1.2. Probability functions for the parallel system. The asterisk * indicates that the correspondent value is less than 0.0001

t	$O_{0,0}(t)$	$O_{0,1}(t)$	$O_{1,0}(t)$	$O_{1,1}(t)$	$O_{0,1_R}(t)$	$O_{1,1_R}(t)$	$O_{1_R,0}(t)$	$O_{1_R,1}(t)$	$O_{1_R,1_R}(t)$
0	1	0	0	0	0	0	0	0	0
10	0.9680	*	0.0001	*	0.0158	*	0.0158	*	0.0003
50	0.8503	0.0075	0.0095	0.0001	0.0644	0.0007	0.0623	0.0005	0.0047
1000	0.4120	0.1602	0.1739	0.0676	0.0682	0.0288	0.0575	0.0223	0.0095
∞	0.4122	0.1601	0.1738	0.0675	0.0682	0.0288	0.0575	0.0223	0.0095

Table 1.3. Performance measures for the parallel system

t	$A_S(t)$	$R_S(t)$	$R_S(t, \tau)$	$\upsilon_S(t)$
0	1.0000	1.0000	0.9531	0.0000
10	0.9997	0.9997	0.9475	0.0000
50	0.9953	0.9939	0.9278	0.0001
1000	0.9905	0.3544	0.9363	0.0002
∞	0.9905	0	0.9363	0.0002

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